Chapter 9
Bet Strategies

9.1 Risk and Capitalization

9.1.1 Risk in a Game with Fixed Return

Blackjack is a game in which the expected return on a given round differs from that of the previous round, because the pack composition has changed; as a result of the differing return, the bet should change correspondingly. A proper analysis of risk in such a game must account for these fluctuations. Nevertheless, analysis of a simplified game in which the return and bet stay fixed is a useful warm-up exercise, in terms of both the mathematical tools employed and some characteristics of the results. The reasoning here, as in much of this chapter, follows Werthamer (2005).

Thus, consider playing a model game consisting of a sequence of rounds, on each of which a fixed amount B is bet. Every round has an outcome in which Player wins an amount equal to his bet with probability $\xi(+1)$, loses his bet with probability $\xi(-1)$, and ties (i.e., no money is exchanged) with probability $\xi(0)$. (Ties are included because they occur in blackjack, as seen in Chap. 2, about 10% of the time; but for now overlook the different payoffs from Player blackjack, doubling, and splitting.) Obviously, since these three possibilities exhaust all outcomes, $\sum_\omega \xi(\omega) = 1$, where $\omega = \pm 1, 0$. In this game, the expected return is $R = \sum_\omega \omega\xi(\omega)$; and the variance of the return is $\sigma^2 = \sum_\omega (\omega - R)^2 \xi(\omega) = 1 - \xi(0) - R^2$.

Assume that Player begins this model game with trip capital C_0, taken for convenience to be an integer multiple of B. The "coverage" is the dimensionless ratio $c_0 \equiv C_0/B$. If Player then plays N rounds where $N < c_0$, he cannot lose all his money even if he loses every round. In that case, it is easy to show (later, from (9.9)) that Player's expected capital becomes $\langle C_N \rangle = C_0 + NBR$, with variance $\langle C_N^2 \rangle - \langle C_N \rangle^2 = N\,B^2\sigma^2$. His expected capital has grown or shrunk, directly proportional to N, depending on whether R is positive or negative.

The more realistic case, $N \geq c_0$, includes situations where Player's trip capital is not big enough, and/or his bet size is too large, and/or he plays too many rounds, to cover every possible losing streak. He may then, with some nonvanishing probability, lose his entire stake and be forced to stop playing. This event is

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traditionally called "ruin." Deriving how Player's capital evolves, taking account of the ruin possibility, is mathematically nontrivial.

Begin that analysis with the standard trinomial expression for $p_N(\mathbf{n})$, the probability of N rounds resulting in a total of n_+ wins, n_- losses, and n_0 ties:

$$p_N\left(\mathbf{n}\right) = N!\prod_{\omega}\left(\xi\left(\omega\right)^{n_\omega}/n_\omega!\right) \tag{9.1}$$

Furthermore, the conditional probability $\hat{p}_N\left(\mathbf{n}\right)$ of that result occurring without ruin (i.e., with "survival") is

$$\hat{p}_N\left(\mathbf{n}\right) = \left(1 - \prod_{\omega}\frac{n_\omega!}{\left(n_\omega + \omega c_0\right)!}\right) p_N\left(\mathbf{n}\right) \tag{9.2}$$

This expression holds for all n_- by interpreting $n_-!/\left(n_- - c_0\right)! = 0$ for $n_- < c_0$. Expression (9.2) can be deduced only with difficulty, but it can readily be proved by induction: if true for $\hat{p}_N$, then it can be shown to be true as well for $\hat{p}_{N+1}$, due to the basic relationship

$$\hat{p}_{N+1}\left(\mathbf{n}\right) = \sum_{\omega}\hat{p}_N(\mathbf{n} - \boldsymbol{\omega})\xi\left(\omega\right) \tag{9.3}$$

More useful for further analysis are the probabilities $\hat{p}_N(C)$ of Player's capital being C after N rounds, conditional on survival, given by

$$\hat{p}_N(C) \equiv \sum_{\mathbf{n}}\delta\left(N, \sum_{\omega}n_\omega\right)\delta\left(C - C_0, B\sum_{\omega}\omega\, n_\omega\right)\hat{p}_N(\mathbf{n}), \tag{9.4}$$

and its ruin-free analog, $p_N(C)$. The former satisfies a recursion like (9.3),

$$\hat{p}_{N+1}(C) = \Theta(C)\sum_{\omega}\hat{p}_N(C - \omega B)\,\xi\left(\omega\right) \tag{9.5}$$

using the step function $\Theta(C) \equiv 1, C > 0;\ \equiv 0, C \leq 0$. These quantities are convenient when deriving the probability L_N of ruin on the Nth round: since ruin occurs when a Player with capital B bets it all and loses, so that $L_N = \hat{p}_{N-1}(B)\xi(-1)$, substitution of (9.1)–(9.4) and some manipulation shows that

$$L_N = \Theta(N - c_0)\left(c_0/N\right)p_N(0) \tag{9.6}$$

Results equivalent to (9.4) are quoted by Epstein (1995, p. 68, Eqs. 3–11) and attributed to Lagrange.

Conservation of probability can be confirmed using (9.5): since

$$\sum_C \hat{p}_N(C) = \sum_C \hat{p}_{N-1}(C) - L_N, \tag{9.7}$$

hence iterating downward $N - c_0$ times gives

$$\sum_C \hat{p}_N(C) + \sum_{\mu=c_0}^{N} L_\mu = 1; \tag{9.8}$$

in words, the probability of survival plus the cumulative probability of ruin is unity. Furthermore, a similar approach shows that the expected capital after N rounds is

$$\begin{aligned}
\langle C_N \rangle &= \sum_C C\ \hat{p}_N(C) = \sum_C C \sum_\omega \hat{p}_{N-1}\,(C - \omega B)\xi\,(\omega) \\
&= \sum_{C'} \sum_\omega \left(C' + \omega B\right) \hat{p}_{N-1}\left(C'\right) \xi\,(\omega) \\
&= \langle C_{N-1} \rangle + B\ R \left(1 - \sum_{\mu=c_0}^{N-1} L_\mu\right) \\
&= C_0 + B\ R \left(N - \sum_{\mu=c_0}^{N} (N - \mu)\ L_\mu\right).
\end{aligned} \tag{9.9}$$

When there is no possibility of ruin, so that $L_\mu = 0$ for all μ, (9.9) proves the assertion regarding the expected capital made earlier.

The results so far have all been exact. But the simplicity of (9.9) is deceptive: after substituting (9.6) and its predecessors (9.4) and (9.1), no further progress can be made toward a simpler closed expression. Instead, resort to taking the limit where the number of wins, losses, and ties becomes large; correspondingly, also regard $N \sim c_0$; the derivation is outlined in Appendix 1. The result is that

$$\begin{aligned}
\frac{\langle C_N \rangle - C_0}{N\ B\ R} \approx 1 - \exp\left(-c_0 \bar{R}\right) \sum_\pm \exp\left(\pm c_0 \hat{R}\right) \left(1 - \frac{c_0^2\,/\,\hat{N}}{c_0 \hat{R}}\right) \\
\times \operatorname{Erf}\left(\frac{c_0^2/\hat{N} \pm c_0 \hat{R}}{\left(c_0^2/\hat{N}\right)^{1/2}}\right),
\end{aligned} \tag{9.10}$$

in terms of the error function (sometimes called the complementary normal integral)

$$\operatorname{Erf}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty dy\ \exp\left(-\frac{y^2}{2}\right); \tag{9.11}$$

a direct integration shows that $\mathrm{Erf}(0) = 1/2$, and hence that $\mathrm{Erf}(-\infty) = 1$. The caret variables in (9.10) (distinct from those of (8.7)) are here defined as $\hat{N} \equiv N\sigma^2$, $\hat{R} \equiv R/\sigma^2$, and $\bar{R} \equiv -\hat{R} + \ln((1+\hat{R})/(1-\hat{R}))$.

The same analytical techniques can be used to derive the cumulative ruin probability,

$$W_N \equiv \sum_{\mu=c_0}^{N} L_\mu = \exp\left(-c_0\bar{R}\right)\sum_{\pm}\exp\left(\pm c_0\hat{R}\right)\mathrm{Erf}\left(\frac{c_0^2/\hat{N} \pm c_0\hat{R}}{\left(c_0^2/\hat{N}\right)^{1/2}}\right) \tag{9.12}$$

as well as a lengthier expression for the variance of the capital. In the very large N limit, $\hat{N} \approx \hat{R}^{-2}$, (9.12) reduces to just $W_N \approx \exp[c_0 \ln((1 - R/\sigma^2)/(1 + R/\sigma^2))]$. The similar but not identical expression, $W_N = \exp[(c_0/\sigma)\ln((1 - R/\sigma)/(1 + R/\sigma))]$, with σ^2 as the exact variance, is quoted without proof both by Carlson (1992, p. 156) and by Schlesinger (2005, p. 112). It appears still earlier as Eq. 9.5 of Sileo (1992), who apparently misuses a model due to Griffin (1999, pp. 141–142). The expression here, however, does agree with Epstein (1995, p. 59). For the remainder of this chapter, assume the return is small, $R \ll 1$, so that this discrepancy disappears and $\bar{R} \cong \hat{R}$.

Comparison of (9.10) and (9.12) shows that $\left(\langle C_N\rangle - C_0\right)/NBR$ is just the probability of survival at N but with a correction term that removes random walks into negative capital prior to N. The right-hand side of (9.10), representing the effect of ruin, is straightforward to evaluate numerically and graph, as is (9.12); although the computation is deferred to the end of the next section, the correction term does not have a major qualitative effect.

These results can also be obtained with a different but complementary mathematical approach, one that will prove valuable in the next section. In this approach, introduce the Fourier representation,

$$\delta(m,n) = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp\left(i\,(m-n)\,\theta\right), \tag{9.13}$$

for each of the Kronecker delta functions in (9.4). Then the sums over $\mathbf{n}$ can easily be performed, leading to the expression

$$p_N(C) = \frac{B}{2\pi}\int_{-\pi/B}^{\pi/B} d\varphi\ \exp\left[-i\,(C - C_0)\,\varphi + N\ln\nu(\varphi)\right], \tag{9.14}$$

where

$$\nu(\varphi) \equiv \sum_{\omega}\xi(\omega)\,\exp\left(i\,\omega B\,\varphi\right). \tag{9.15}$$

Furthermore, (9.15) can now be generalized to include the differing payoffs of doubles, splits, and blackjack by extending the definition of ω to also include values ± 2 and 3/2,

$$\nu(\varphi) \to \sum_{\omega=0,\pm1,\pm2,3/2} \xi(\omega)\,\exp(i\,\omega B\,\varphi)\,. \tag{9.16}$$

The asymptotic limit of $p_N(C)$ for $N \gg c_0$ can be derived by applying the Stationary Phase method to the φ integration. The integrand is asymptotically vanishing except near the zero of $\ln \nu$, which is at $\varphi = 0$. Thus,

$$\ln \nu(\varphi) \approx iBR\varphi - B^2\sigma^2\varphi^2/2, \tag{9.17}$$

which in turn, after extending the integration limits to $\varphi = \pm\infty$, leads to

$$p_N(C) \approx \left(2\pi\hat{N}\right)^{-1/2} \exp\left(-\left(C - C_0 - NBR\right)^2/2\hat{N}B^2\right); \tag{9.18}$$

the caret variables now are defined to contain the exact variance of the return. Player's capital is Gaussian distributed, with mean $C_0 + NBR$ and width $\hat{N}^{1/2}B$, just as for finite N. Substituting (9.18) back into (9.6) and (9.9) reproduces (9.10) and (9.12).

However, the Fourier representation is unsuitable for the probability $\hat{p}_N(C)$ conditional on survival. For example, (9.10) cannot be reproduced directly without going through (9.6). This obstacle complicates the generalization to nonvanishing risk of ruin. Of value, though, is having found the point of stationary phase in (9.15) to be at $\varphi = 0$.

9.1.2 Optimal Betting When Return Fluctuates: Expected Capital and Risk

The instructive but simplified warm-ups of the previous section facilitate taking on the real game of blackjack, where the return R varies from round to round. Section 8.1 has already demonstrated that R is distributed about a mean, $\langle R\rangle$, with a width that increases with the fraction of the pack dealt out. For a game with multiple decks, $\langle R\rangle$ is negative. That part of the distribution where $R > 0$ offers the opportunity to increase the bet size, so that $B \to B(R)$. These features are illustrated in Fig. 9.1.

Because each round has a different return and bet size, label the rounds by an index, μ, and generalize the quantities of the previous section to $R_\mu, B_\mu, \nu_\mu(\varphi)$. The bet B_μ is some integer multiple of a base bet B_0. Then the probability distribution $p_N(C)$ is naturally expressed in the Fourier representation, generalizing (9.14), as

$$p_N(C) = \frac{B_0}{2\pi}\int_{-\pi/B_0}^{\pi/B_0} d\varphi\ \exp\left(-i\,(C - C_0)\,\varphi + \sum_{\mu=1}^{N} \ln \nu_\mu(\varphi)\right) \tag{9.19}$$

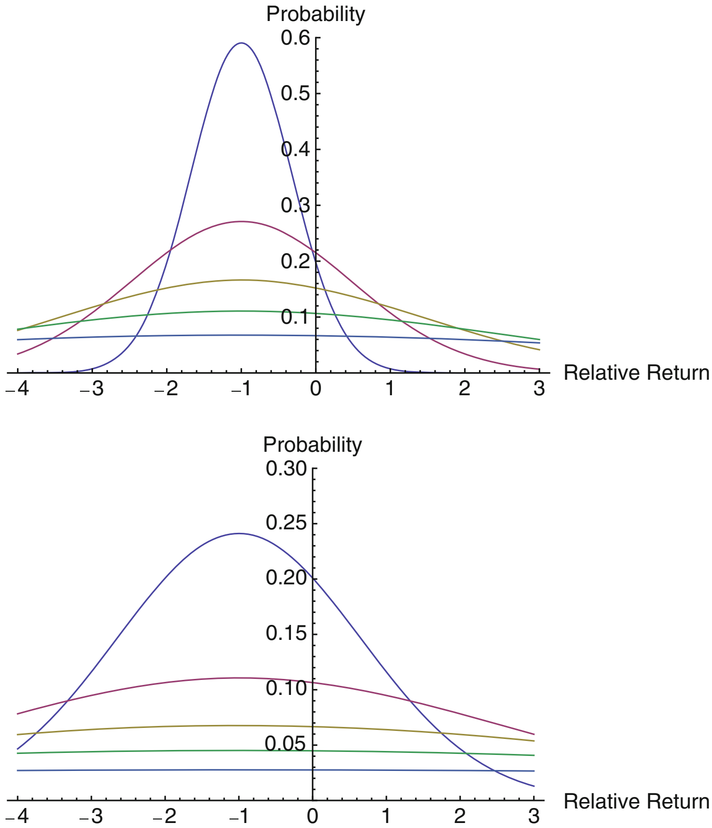

Fig. 9.1 Distribution of return, scaled to its magnitude at zero count, for various depths: *upper* plot for six decks, *lower* plot for one deck. At small depths the distributions are narrowly peaked around −1, while as the depth grows they spread further out and have increasing weight at positive returns. The spreading is more pronounced for one deck than for six

Once again, temporarily disregard the distinctive payoffs to hands that are doubled, split, or blackjack.

For a sufficiently long sequence of rounds, through multiple reshuffles of the shoe, the sum over μ tends to its ensemble average. Then, in (9.19),

$$\sum_{\mu=1}^{N} \ln v_{\mu}(\varphi) \approx N \left\langle\left\langle v(\varphi)\right\rangle\right\rangle \approx \left\langle\left\langle \hat{N}\left(i\ B\hat{R}\varphi - B^2\varphi^2/2\right)\right\rangle\right\rangle \tag{9.20}$$

the second equivalence in (9.20) takes the small-argument expansion of $v(\varphi)$, appropriate for the asymptotic, large N limit. The φ integral, just as in the derivation of (9.18), then gives

$$p_N(C) = \frac{B_0}{\sqrt{2\pi \left\langle\!\left\langle \hat{N} B^2 \right\rangle\!\right\rangle}} \exp\left[-\frac{(C - C_0 - N \langle\langle BR \rangle\rangle)^2}{2 \left\langle\!\left\langle \hat{N} B^2 \right\rangle\!\right\rangle} \right]. \tag{9.21}$$

Comparison of (9.18) and (9.21) shows that when the return and bet size vary, the expected capital after N rounds becomes $\langle C_N \rangle = C_0 + N \langle\langle BR \rangle\rangle$. It grows with N to the extent that $\langle\langle BR \rangle\rangle > 0$, even if $\langle R \rangle < 0$. Note that $\langle\langle BR \rangle\rangle$ is just the (risk-free) yield per round, Y, central to Chaps. 3 and 4. Thus, a desirable goal for bet strategy is to arrive at a functional dependence of B on R for which the yield is positive, so that the expected capital grows rather than shrinks with N.

The analysis of this section has thus far neglected ruin, thus omitting the key source of risk. To include this effect, return to (9.5) and generalize it to

$$\hat{p}_N(C) = \Theta(C) \sum_{\omega} \hat{p}_{N-1} (C - \omega B_N)\, \xi_N(\omega). \tag{9.22}$$

Also, the ruin probability becomes $L_N = \hat{p}_{N-1}(B_N)\, \xi_N(-1)$; and $L_N = 0$ if $N < c$, where the coverage c is now defined via

$$C_0 = \sum_{\mu=1}^{c} B_\mu \equiv c \langle\langle B \rangle\rangle. \tag{9.23}$$

A related coverage parameter that will be especially useful later is $\bar{c} \equiv C_0 / \left\langle\!\left\langle B^2 \sigma^2 \right\rangle\!\right\rangle^{1/2}$, the ratio of trip capital to what in Chap. 4 is called the typical bet. Here, "typical" means the root-mean-square total amount wagered on a hand, including the increased initial bets on hands with positive expected return and the additional amounts needed to cover doubles and splits.

The analysis leading to (9.8) and (9.9) proceeds as before: probability is conserved, just as expressed by (9.8), and the expected capital becomes

$$\langle C_N \rangle = C_0 + \sum_{\mu=1}^{N} B_\mu R_\mu - \sum_{\mu=1}^{N} L_\mu \sum_{\nu=\mu+1}^{N} B_\nu R_\nu. \tag{9.24}$$

For large N, the sum $\sum_\nu B_\nu R_\nu$ includes enough terms that it approaches its ensemble average; furthermore, it is uncorrelated from the outcome of all rounds $\nu \le \mu$. Hence

$$\langle C_N \rangle = C_0 + \langle\langle BR \rangle\rangle \left(N - \sum_{\mu=c}^{N} (N - \mu)\, L_\mu \right). \tag{9.25}$$

To proceed further, a generalization of (9.6) is needed; it could then be combined with (9.21) and (9.25) to derive a generalization of (9.10). To accomplish this, first rederive (9.6) by iteration of $L_N = \hat{p}_{N-1}(B)\, \xi(-1)$, but instead use (9.3) and avoid (9.2). The result, in schematic form, is

$$L_N = \sum_{\text{perms}} \delta\left(-C_0, B\sum_{\omega} \omega\, \xi(\omega)\right) \prod_{\omega} \xi(\omega)^{n_\omega}, \tag{9.26}$$

where the sum is over all permutations of sequences in which the capital goes from C_0 to 0 in N rounds without reaching 0 at any intermediate round. But from (9.6), when the probability factors are the same for every hand,

$$\sum_{\text{perms}} \cdots = \frac{C}{N} N! \sum_{\mathrm{n}} \delta\left(N, \sum_{\omega} n_\omega\right) \prod_{\omega} \left(\frac{1}{n_\omega!}\right) \cdots \tag{9.27}$$

By introducing the Fourier representation for the delta function, (9.26) can be expressed as

$$L_N = \frac{B}{2\pi} \int_{-\pi B}^{\pi B} d\varphi \exp(iC_0\varphi) \sum_{\text{perms}} \prod_{\omega} (\xi(\omega) \exp(i\,\omega B\varphi))^{n_\omega}. \tag{9.28}$$

When the returns and bets vary from round to round, the form (9.28) for L_N generalizes to

$$L_N = \frac{B_0}{2\pi} \int_{-\pi/B_0}^{\pi/B_0} d\,\varphi\ \exp(iC_0\varphi) \sum_{\text{perms}} \prod_{\omega} \prod_{\mu_\omega} \xi_{\mu_\omega}(\omega) \exp\left(i\,\omega B_{\mu_\omega}\varphi\right), \tag{9.29}$$

where the products over the μ_ω respectively contain n_ω terms. Again, as for (9.17), each product contains enough terms, when N is large, to tend to its ensemble average and is uncorrelated from the outcomes of the other sequences. Then

$$\begin{aligned} L_N &\approx \frac{B_0}{2\pi} \int_{-\pi/B_0}^{\pi/B_0} d\,\varphi \exp(iC_0\varphi) \sum_{\text{perms}} \prod_{\omega} \langle\langle \xi(\omega) \exp(i\,\omega B\varphi)\rangle\rangle^{n_\omega} \\ &\approx \frac{B_0}{2\pi} \frac{C}{N} \int_{-\pi/B_0}^{\pi/B_0} d\,\varphi \exp(iC_0\varphi + N \ln\langle\langle v(\varphi)\rangle\rangle). \end{aligned} \tag{9.30}$$

The second line follows by using (9.27) and the argument leading to (9.14).

The last step in the derivation points out that

$$\begin{aligned} N \ln\langle\langle v(\varphi)\rangle\rangle &\approx \left\langle\!\left\langle \hat{N}\left(i\,B\hat{R}\varphi - B^2\varphi^2/2\right)\right\rangle\!\right\rangle \\ &\approx N\langle\langle \ln v(\varphi)\rangle\rangle \end{aligned} \tag{9.31}$$

in the asymptotic limit, using $\mid B\varphi \mid \ll 1$ and the variance of the yield $\ll$ the mean square bet size. Thus (9.30) and (9.31), compared with (9.19)–(9.21), show again that $L_N \approx \Theta(N-c)\,(c/N)\;p_N(0)$ as anticipated. Substituting into (9.25) leads to a form for $\langle C_N\rangle$ just like (9.10), but with the replacements $BR \to \langle\langle BR\rangle\rangle$, $c_0^2/\hat{N} \to C_0^2/N\,\langle\langle B^2\sigma^2\rangle\rangle = \bar{c}^2/N$, and $c_0\hat{R} \to \langle\langle BR\rangle\rangle\,C_0/\langle\langle B^2\sigma^2\rangle\rangle \equiv q$: explicitly,

$$\begin{aligned}\frac{\langle C_N\rangle - C_0}{N\,\langle\langle BR\rangle\rangle} &= 1-\exp(-q)\sum_{\pm}\exp(\pm q)\left(1\pm\frac{\bar{c}^2/N}{q}\right)\\ &\qquad\times\mathrm{Erf}\left(\frac{\left(\bar{c}^2/N\right)\pm q}{\left(\bar{c}^2/N\right)^{1/2}}\right)\\ &\equiv \Psi_N.\end{aligned} \tag{9.32}$$

Ψ_N, of course, is just the yield reduction factor (YRF) discussed in Sect. 4.1. Adding plausibility to this result for $\langle C_N\rangle$ is its close parallel to the generalization that takes, in the absence of ruin, the distribution of C from (9.18) for fixed betting to (9.21) for variable betting. Equation (9.12), for the cumulative ruin probability, similarly generalizes to

$$W_N \to \exp(-q)\sum_{\pm}\exp(\pm q)\,\mathrm{Erf}\left(\frac{\left(\bar{c}^2/N\right)\pm q}{\left(\bar{c}^2/N\right)^{1/2}}\right). \tag{9.33}$$

Schlesinger (2005, p. 132) displays, without proof but attributed (private communication) to Chris Cummings, an expression equivalent to (9.33).

9.1.3 Connections with Finance

Although expression (9.33) is relatively unfamiliar in blackjack analysis, it is similar to a celebrated milestone in the mathematics of finance: the Black–Scholes formula for the rational price of an option on an asset, such as a stock, whose price fluctuates. This result, derived by Fischer Black and Myron Scholes (1973), and by Robert Merton (1973), earned the Nobel Prize in Economics for Scholes and Merton (Black had died before the prize was awarded). Finding such a close connection between a game like blackjack on the one hand, where the outcome of each round is statistically independent of all previous rounds (i.e., a Markov stochastic process), and the stock market on the other requires the observation, which Black, Scholes, and Merton (BSM) adopted from earlier work of Samuelson (1965), that fluctuations of the logarithm of stock prices approximately obey Markov statistics.

BSM take a different mathematical path to their result than here, where the rounds of play were first regarded as discrete, arriving at expressions (9.2) and (9.9), and then the asymptotic, or continuum, limit was taken to reach (9.10) and (9.12). BSM instead assume the continuum limit at the start of their derivation, appeal to the stochastic calculus of Ito (1951), and obtain their formula as the solution of a drifting diffusion equation. Their approach is plausible since a Markov process generates a random walk, whose continuum limit is diffusion. The connection with the approach here can be confirmed directly by verifying that the expected capital, (9.32), satisfies the standard diffusion equation,

$$\left(\frac{\partial}{\partial N} - \langle\langle BR\rangle\rangle \frac{\partial}{\partial C_0} - \frac{1}{2}\left\langle\!\left\langle B^2\sigma^2\right\rangle\!\right\rangle \frac{\partial^2}{\partial C_0^2}\right)\langle C_N\rangle = 0 \tag{9.34}$$

– as does the cumulative ruin probability, (9.33), as well – with diffusion constant $\left\langle\!\left\langle B^2\sigma^2\right\rangle\!\right\rangle$ and drift, or bias, $\langle\langle BR\rangle\rangle$; the number of rounds is a proxy for time. Skipping the discrete model in favor of the continuum could have led to (9.32), for instance, just by solving the diffusion equation with the conditions $\langle C_{N\to 0}\rangle \to C_0$, $\langle C_N\rangle = 0$ for $C_0 = 0$ and $N > 0$. However, since the approach here only requires the Stirling approximation and a steepest descents integration, (9.32)–(9.33) can be conjectured (see below) to already be close approximations for $N \approx 50$, not nearly so large as the true continuum limit.

An approach to deriving the Black–Scholes result based on discrete steps, as here, was later taken by Cox, Ross, and Rubenstein (Cox et al. 1979, CRR). Their rationale was to circumvent the sophisticated continuum mathematics of the stochastic calculus. They start with a simple binary model in which equity prices fluctuate either up or down by a fixed amount (like here, but without ties), such that the log of the price conforms to a Markov process. Then they take the limit of increasingly short time steps and small fluctuation amplitudes. Since this asymptotic limit, applied in their case to the value of an option, involves only the mean and variance of the Markov process, they are able to argue that the result would also hold for any more complex process, such as the actual stock market, provided only that the mean and variance of the model are matched exactly with those of the market. Their argument is an alternative way to derive a diffusion equation like (9.34) whose coefficients are averages over the actual process – blackjack in our application, equity markets in theirs. The CRR approach thus provides a direct proof of (9.32), alternative to the argument given here. Furthermore, subsequent simulations with the CRR model, e.g., Hull (2002), have demonstrated that the continuum limit of the discrete step model is approached quite closely in well under a 100 steps, bolstering the conjecture above.

9.1.4 Properties of the Risk and Expected Capital Expressions

Having confirmed that (9.32)–(9.33) are the appropriate generalization of (9.10) and (9.12) for varying returns and bets, they can now be analyzed in more detail. The expected capital and ruin probability depend on the two parameters, $N/\bar{c}^2$ and q, each of which can range from 0 to ∞; the expressions also depend on the sign of $Y = \langle\langle BR\rangle\rangle$, although here a bet scheme with positive yield is assumed. Numerical evaluations of (9.32)–(9.33) are straightforward. The survival probability, $1 - W_N$, and the YRF, Ψ_N, are plotted in Fig. 9.2 on a log–log scale. Although the survival

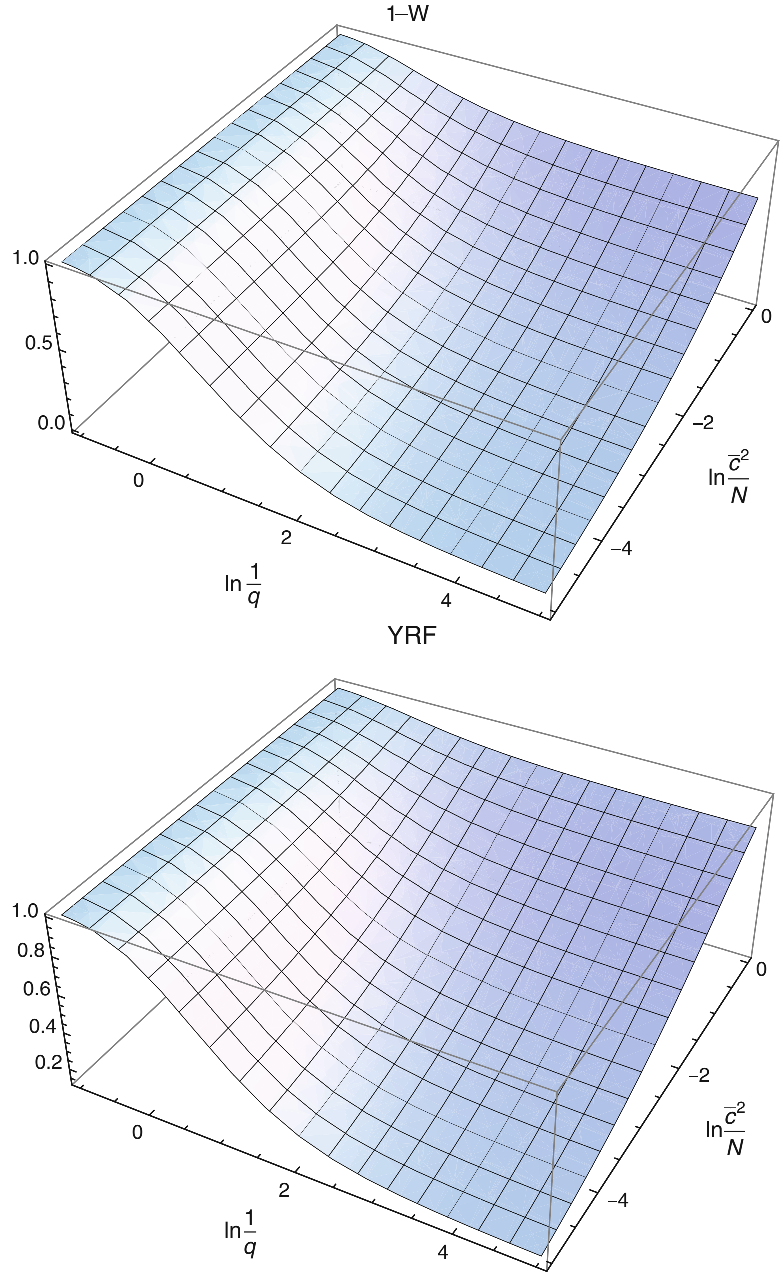

Fig. 9.2 Survival probability (**a**) and yield reduction factor (**b**) as functions of the parameters q and N, log–log scale

probability is not precisely identical to the YRF, the two are qualitatively similar and differ by at most a factor of two.

Figure 9.2 illustrates that the YRF and survival probability vary monotonically with q (i.e., always increase as q increases, and vice versa), so q is called the "ruin protection" in Sect. 4.1. When q is large enough (e.g., comparable to or larger than 1), ruin is very unlikely and the YRF approaches unity for all but extraordinarily large values of $N/\bar{c}^2$(as shown in the left-hand portion of the plot). But for q significantly less than one, the YRF decreases steeply with increasing $N/\bar{c}^2$ (right-hand portion), especially after many rounds such that $N \gg \bar{c}^2$. The conclusion is that ruin can be a decided risk if the coverage is insufficient.

A way to understand these results more intuitively is to look back at (9.21) for the probability distribution of capital, C. The peak of the distribution drifts linearly with increasing N to larger positive values, while the width increases as $N^{1/2}$. Ruin becomes likely to the extent that the distribution has any significant weight at $C = 0$, i.e., when $(C_0 + N\langle\langle BR\rangle\rangle)^2 < N\langle\langle B^2\sigma^2\rangle\rangle$. This condition, in turn, is satisfied if the two conditions $q < \bar{c}/N^{1/2} < 1$ both hold, much like the conditions (low ruin protection and low capital coverage) of the previous paragraph (the foreground of Fig. 9.2). Ruin occurs when the distribution, despite the steady drift of its peak to more positive values, has diffused outward to overlap $C = 0$ (see Fig. 9.3). Ruin protection affects the relation of the drift to the broadening; large ruin protection q, as well as large coverage $\bar{c}$, keeps the overlap exponentially small and forestalls ruin.

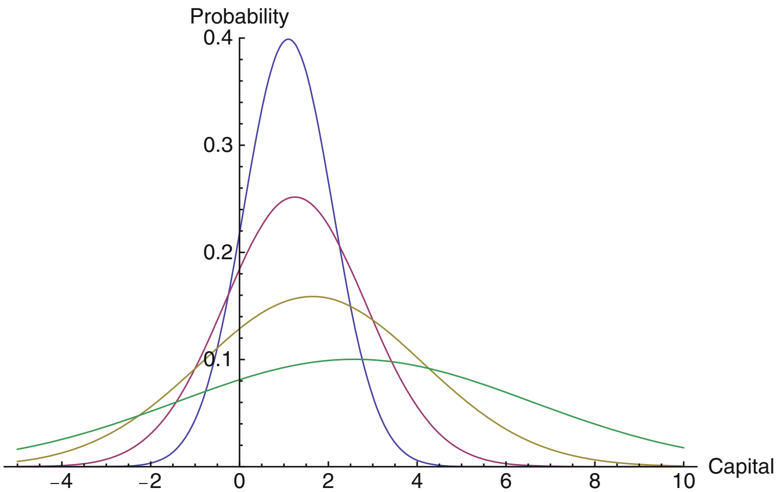

Fig. 9.3 Representation of the distribution of capital, for several values of the number of rounds N. As N increases, the distribution shifts, widens, and flattens: its mean and its variance both increase

9.1.5 Optimal Betting When Return Fluctuates: Bet Strategy

Having derived the evolution of the expected capital with the relative number of rounds played, dependent also on the ruin protection, the bet strategy can now be optimized. The criterion for optimization invoked here is to find the bet function that maximizes the expected capital for a given number of rounds, yet fixes the cumulative ruin probability at a predetermined value. Microeconomists might instead prefer the introduction and maximization of a Utility Function (see, e.g., Ingersoll (1987), especially Chap. 1), but such a criterion is primarily a formal one, difficult to apply for any practical results.

Thus, functionally maximize $\langle C_N \rangle$ – or more conveniently an "effective yield" $y_N \equiv (\langle C_N \rangle - C_0)/NC_0 = Y\,\Psi_N/C_0$ – with respect to $B(\gamma)$ while holding W_N constant.

However, $B(\gamma)$ is also constrained to lie within the range between the table minimum B_- and maximum B_+ as set by house rules. If Player elects, for whatever reason, to cap his bet size below the table maximum, then B_+ instead represents that cap. Similarly, he might also choose his base bet, B_-, above the table minimum, as long as $B_- \leq B_+$. In any event,

$$\langle\langle BR \rangle\rangle = \left(B_- \int_{-\infty}^{\gamma_-} + B_+ \int_{\gamma_+}^{\infty} \right) d\gamma\, \rho(\gamma)\, R(\gamma) \tag{9.35a}$$
$$+ \int_{\gamma_-}^{\gamma_+} d\gamma\, \rho(\gamma)\, B(\gamma)\, R(\gamma)$$

and a similar expression for $\left\langle\left\langle B^2\sigma^2 \right\rangle\right\rangle$, referenced later as (9.35b). Since $B(\gamma)$ is intuitively expected to increase monotonically with γ, define $\gamma_\pm$ via $B(\gamma_\pm) = B_\pm$. The "spread," then, is $\beta \equiv B_+/B_-$.

The maximization of y_N with respect to $B(\gamma)$ in the range $\gamma_- \leq \gamma \leq \gamma_+$ – with the constraint of fixed W_N best handled by use of a Lagrange multiplier, denoted L – leads to the functional stationary equation $\delta\,(y_N - LW_N)/\delta B(\gamma) = 0$. The differentiation at first looks complicated; but the complexity can be circumvented by recognizing, from (9.32) and (9.33), that the expression depends on $B(\gamma)$ only via $\langle\langle BR \rangle\rangle$, whose functional derivative is proportional to $R(\gamma)$, and on $\left\langle\left\langle B^2\sigma^2 \right\rangle\right\rangle$, whose functional derivative is proportional to $B(\gamma)\,\sigma^2$. Thus, the result of differentiation is just a linear combination of these two simple expressions, and so the stationary equation has the solution $B(\gamma) \propto R(\gamma)/\sigma^2$. As expected, the bet is proportional to R between a lower and upper threshold. Define a constant of proportionality s such that $b\,(\gamma) \equiv B(\gamma)/C_0 = s\,R(\gamma)/\sigma^2$ in this range. Then the threshold true counts satisfy $R(\gamma_\pm) = b_\pm\,\sigma_\pm^2/s$. A bet function of this form is consistent with an existence theorem asserted by Epstein (1995, p. 67), and referenced by Griffin (1998,

p. 139). Substitution of this functional form for $b(\gamma)$ into (9.35a) and (9.35b) leads to the expressions

$$\langle\langle bR\rangle\rangle = A_1 + sA_2, \qquad \left\langle\!\left\langle b^2\sigma^2\right\rangle\!\right\rangle = A_0 + s^2 A_2, \tag{9.36}$$

where the A coefficients are defined as weighted integrals over γ.

Although the expected return in general is a nonlinear function of true count – Sect. 8.2 has shown it can be closely approximated by a cubic – for purposes of this section we adopt the rough approximation that the function is linear, and hence is independent of depth. In this event, the distribution of returns is Gaussian just like that of true counts. Then we can convert the $A_{0,1,2}$ quantities into the integrals over R,

$$\begin{aligned}
A_0 &\equiv \left(b_-^2 \int_{-\infty}^{R_-} + b_+^2 \int_{R_+}^{\infty} \right) dR\, \sigma^2 \rho(R), \\
A_1 &\equiv \left(b_- \int_{-\infty}^{R_-} + b_+ \int_{R_+}^{\infty} \right) dR\, R\, \rho(R), \\
A_2 &\equiv \int_{R_-}^{R_+} dR \left(\frac{R}{\sigma}\right)^2 \rho(R),
\end{aligned} \tag{9.37}$$

with weight $\rho(R)$ which incorporates the average of that Gaussian distribution over all depths f of the pack up to the reshuffle penetration F:

$$\rho(R) = \frac{1}{F} \int_0^F df \frac{1}{\sqrt{2\pi}\Delta_R} \exp\left[-\frac{1}{2} \left(\frac{R - R_0}{\Delta_R} \right)^2 \right], \quad \Delta_R \equiv 52\Delta\, R_0. \tag{9.38a}$$

The slope parameter s is set by the specified level of risk, W_N, which in turn is parameterized by q and $\bar{c}$. In principle, it would be desirable to invert that relationship to find s as a function of W_N, and substitute into y_N. But the inversion is not possible in closed form, instead requiring computation. Needing to specify the two bet threshold parameters $b_\pm$ creates additional complexity for a numerical approach.

To gain some partial insights, begin the computational work by choosing the representative set of parameters for a blackjack game adopted in Chap. 4: six decks with penetration at 80% of the shoe before reshuffle; Generic Strategy expected return of -0.005, with variance of 1.26 independent of R; and zero expected return at a true count of $+1$. With these parameters, the distribution of returns becomes

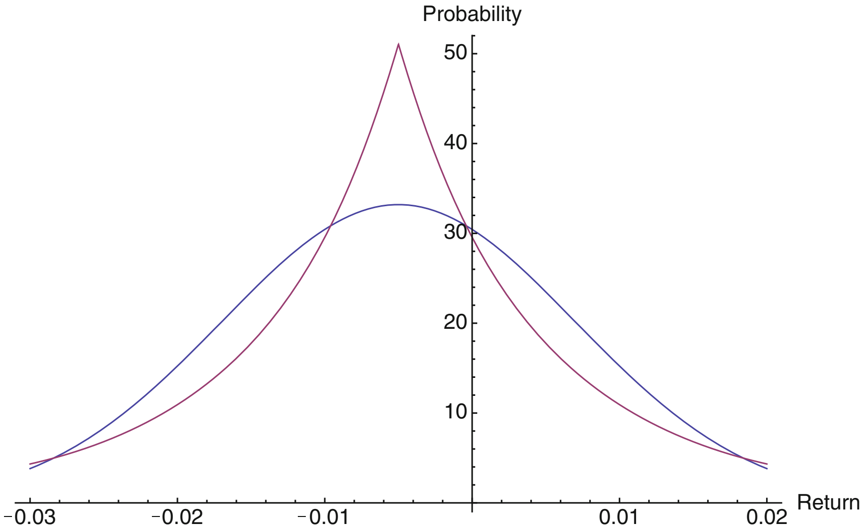

Fig. 9.4 Distributions of expected returns: at depth 0.4 (*Gaussian curve*) and averaged over depths to a penetration 0.8 (*peaked curve*), for representative game parameters

$$\rho(R) = \frac{1}{0.8}\int_0^{0.8} df \frac{1}{\sqrt{2\pi}\Delta_R} \exp\left(-\frac{1}{2}\left(\frac{R+.005}{\Delta_R}\right)^2\right),$$

$$\Delta_R^2 = \frac{52\,(.005)^2 f}{6(1-f)}. \tag{9.38b}$$

Figure 9.4 graphs $\rho(R)$, along with its integrand at $f = 0.4$, showing how the averaging over depth distorts the shape away from a Gaussian. Selecting values for $b_\pm$ and N permits computation of both y_N and W_N vs. the slope s. An example is shown in Fig. 4.2 with the Lifetimer parameters, $b_- = 0.001, b_+ = 0.010$, and $N = 10^6$. Each curve has an extremum: y_N has a shallow maximum while, at a smaller value of slope, W_N has a deep minimum. This pattern is a general characteristic over a wide range of parameters, although the W_N minimum becomes quite shallow for smaller N. Thus, slopes that are either greater than at the maximum, or less than at the minimum, correspond to both lesser yield and greater risk than at the respective extrema; only slopes between these two points correspond to solutions of maximal yield.

With these insights, the slope can be eliminated and y_N obtained directly as a function of W_N, as plotted in Fig. 4.3; the Weekender example, $b_- = 0.01$, $b_+ = 0.10$, $N = 10^3$, is also shown. The latter might be typical of a weekend casino trip, with trip capital of a 100 base bets; the former of a "lifetime" of extensive play, with trip capital of a 1,000 base bets. Also note that the effective yield may in fact be negative for sufficiently low risks, although always greater than that without counting.

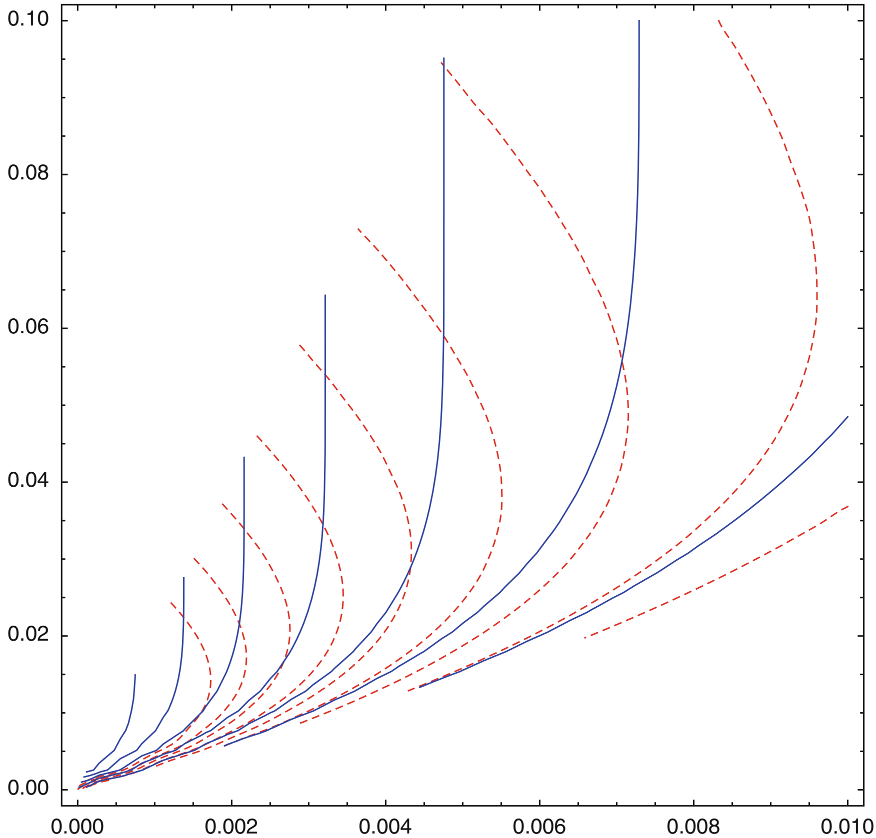

Fig. 9.5 Contours of minimum risk (*solid curves*) and maximum risk (*dashed curves*) vs. $b_{\pm}$

Details of the effective yield vs. risk curves vary with the choice of parameters, although the family of curves retains the general shape shown here. Displaying these variations in full detail is difficult, since the parameters form a three-dimensional family. Nevertheless, some insights can be gained from contour plots over the two-dimensional $b_{\pm}$ plane. First, Fig. 9.5 shows the end points, of greatest and least risk, for the Lifetimer example. Then, Fig. 9.6 shows the contours of effective yield, with the risk chosen at the so-called Kelly value, $W_N = w_K \equiv e^{-2} = .1353$. These contours are bounded by the curves along which w_K intersects the maximum and minimum risk surfaces.

So far, the upper and lower bet bounds are regarded as chosen independently: the lower bound is held at the table minimum with the upper bound as large as practicable. Indeed, Fig. 9.6 shows for a fixed upper bound that y_N decreases as the lower bound is raised; and that for fixed lower bound y_N increases with increasing upper bound. But an emphasis on camouflage and on maintaining a predefined spread suggests instead fixing the *ratio*, β, of upper to lower bounds. Then the geometric

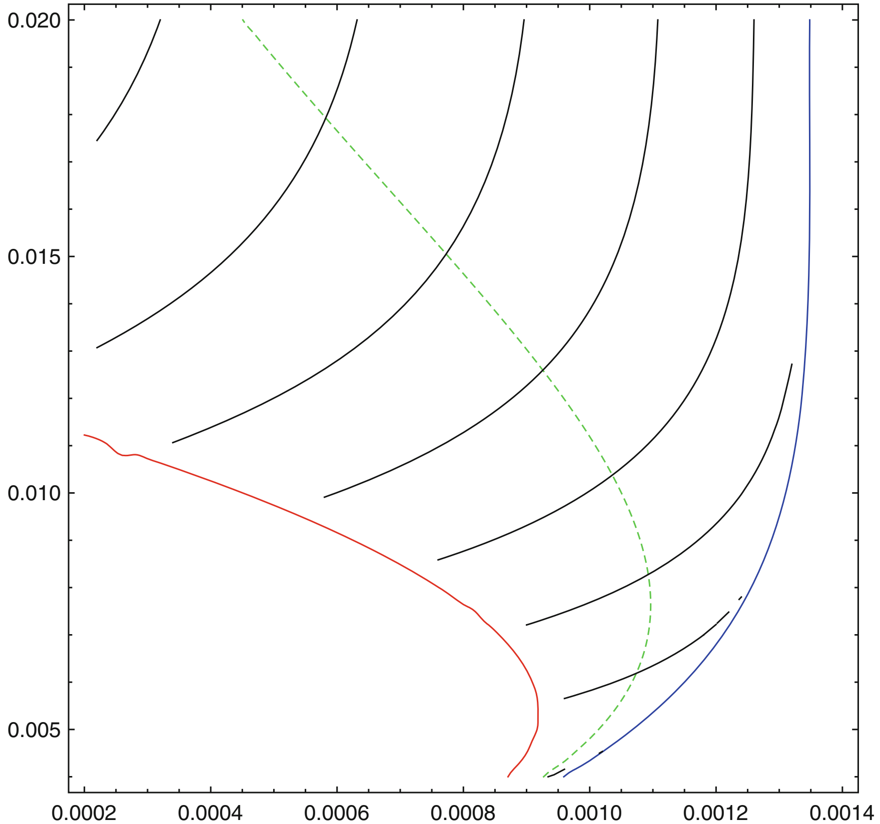

Fig. 9.6 Contours of effective yield vs. $b_{\pm}$, bounded between curves of minimum risk (on the *right*) and maximum risk (on the *left*). The curve of HJY points is *dashed*.

mean of the bounds could be chosen so as to maximize y_N, even if the resulting lower bound exceeds the table minimum. The maximum of y_N under these circumstances occurs for the remarkably simple condition $sq = 1$, which implicitly determines s. Even more remarkably, the maximal condition is independent of the functional forms of y_N and W_N and so, in particular, is independent of N!

This condition was first emphasized by Harris, Janecek, and Yamashita (1997, HJY). But HJY arrive at it by adopting quite different optimization criteria than here. Harris (1997), following Sileo (1992) and Schlesinger (2005), instead posits that a betting optimum maximizes the dimensionless ratio $\langle\langle BR\rangle\rangle^2 / \langle\langle B^2\sigma^2\rangle\rangle$ at fixed spread (actually, yet equivalently, Harris (1997) minimizes the inverse ratio); risk is not a determinant in his criterion. The square root of this ratio is just the ROI (the yield per unit of typical bet size) from Sects. 4.1 and 6.1; maximal return on investment is a natural criterion for a betting optimum. Computed solutions of the HJY condition, also plotted in Fig. 9.6, are a curve through the contours of effective yield. The fact that a ray through the origin of Fig. 9.6 (a line of constant spread) is tangent to a contour at the HJY point provides a visible geometric verification that the point maximizes effective yield for that spread.

But Fig. 9.6 also shows that a continuum of optimal bet patterns, depending on risk, exists in addition to the HJY point. They range from the smallest risk and effective yield, which might be the choice of a casual, lightly capitalized player over a short session, up to the largest risk and effective yield, which could be the choice of a dedicated, well-capitalized player tolerant of occasional ruined sessions to gain greater long-term winnings. But neither end of the continuum is robust: at the lower end, a small increase in risk produces a much larger increase in effective yield; and at the upper end a small decrease in effective yield produces a much larger decrease in risk. A better balance of effective yield and risk occurs at intermediate points.

Since Player would ideally like to both minimize risk and maximize effective yield, one way to make this tradeoff is to maximize the ratio y_N/W_N, the effective yield per unit of risk. Although difficult to pursue analytically, the maximization is straightforward computationally and leads to a well-defined intermediate point, distinct from that of HJY; call it the "minmax," or "MM," point. Whereas the HJY point is at a rather high effective yield as well as risk, the MM point typically occurs at much lower values of both. Thus, the former may be an appropriate choice for the dedicated player, the Lifetimer example of Chap. 4, while the latter seems more suitable for the casual player, the Weekender example.

9.1.6 Yield When the Bet Size Is Discrete; Wong Benchmark Betting

The optimal bet strategy derived in the previous section calls for a bet size that is a ramp function of the expected return. Clearly, however, bets in actual practice must be sized as a discrete multiple (an integer, or half-integer if the casino has chips of that denomination) of the minimum bet. Thus, the ramp of (9.35) must be approximated by a staircase, much like that represented in Fig. 4.1.

Although staircases with various numbers of steps and sizes can be used, their results were asserted earlier to be quite close to that of the corresponding ramp. To model this, select the Benchmark game and a bet spread of 10, and then replace the optimal ramp by a coarse staircase with just two intermediate steps: at 4 and at 7 base bets. Computational results of the performance of the staircase vs. the ramp, for both the Lifetimer (HJY point) and the Weekender (MM point), are displayed in Table 9.1 (the ramp results are the same as in Table 4.1). Comparison demonstrates that both yield and risk are degraded by only a few percent. Other staircases, subdivided more finely, perform even more closely to the ramp.

A betting scheme recommended by Wong (1994, p. 18) and shown in Fig. 9.7 is also a staircase, but it lowers the bet another step for negative true counts and so is suboptimal. Furthermore, Wong specifies the slope of his staircase independently of

Table 9.1 Performance of a coarse staircase vs. a ramp

Player	Bet shape	Steepness	Risk of ruin	Effective yield ratio
Lifetimer (HJY)	Ramp	0.94	0.1185	+0.01716
	Staircase	0.94	0.1213	+0.01713
Weekender (MM)	Ramp	2.80	0.1360	+0.00617
	Staircase	2.80	0.1404	+0.00603

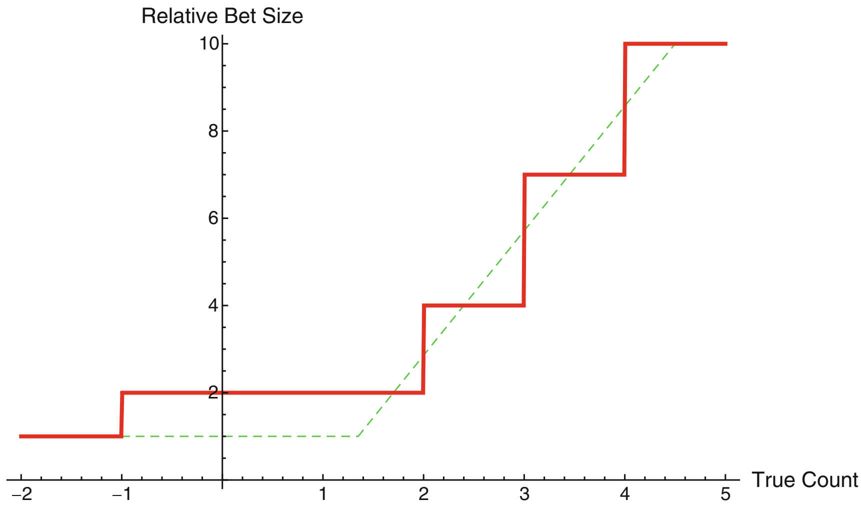

Fig. 9.7 Wong's "Benchmark" bet staircase (*solid*) and ramp modification (*dashed*)

both the penetration and Player's risk profile. Figure 9.7 also shows a ramp function (dashed) that gives a good representation of his staircase.

Wong applies this betting scheme in a computer simulation of his Benchmark game. He quotes a yield ratio of 0.016 and a ratio of average to minimum bet of 2.65. Wong does not exhibit a risk result, possibly because his simulation may assume infinite initial capital so that ruin is never possible. Current commercial simulation programs, however, do permit specification of an initial capital and generate ruin probabilities.

Performance measures for Wong's bet function computed in the Benchmark game are shown in Table 9.2.

The yield ratio is 0.017, with an average bet of 2.70 units, not far from the values he quotes; thus, the simulation and analysis approaches give results reasonably close to each other. Also, with a typical bet of 4.2 units the return on investment is around 0.004. However, modifying Wong's function so that the minimum is bet for true counts less than +2, rather than below 0 as he does, improves both yield and ROI: a hand with negative expected return should optimally not be bet at above the minimum. In addition, Wong's yields fall sharply as the penetration is reduced, becoming negative for penetrations less than 0.5; these low penetrations need an even steeper staircase to sustain a positive yield in such unfavorable circumstances.

Table 9.2 Performance of Wong bet function

Betting style	Yield ratio	ROI	Effective yield	Risk
Wong	0.0173	0.0041		
Wong (modified)	0.0177	0.0045		
Lifetimer			0.016	0.10
Weekender			0.014	0.37

The effective yield and risk, with the ramp shown in Fig. 9.7, for both the Lifetimer and the Weekender, are also listed in Table 9.2. The effective yield is below that of the yield ratio because the former averages over sessions ending in ruin, whose risk is particularly high for the Weekender. The ramp is quite steep; it gives performance close to that of the HJY point: fine for the Lifetimer but probably too aggressive for the Weekender.

9.2 Betting Proportional to Current Capital

9.2.1 One Hand per Round

Now consider bet sizes scaled to the current capital, as first suggested by Kelly (1956). Thus, set $B_N = \varphi C_N$ at the Nth round and apply a criterion for choosing the scaling factor φ, independent of N. In this case, the capital changes multiplicatively rather than additively: $C_{N+1} = C_N\,(1 + \varphi\Omega_{N+1})$, depending on the stochastic process Ω whose values give the amount Player wins on a hand. For a simplified game with just a single expected return, $R = \langle\Omega\rangle > 0$ independent of N, it is easy to show that the expected, or mean, value of C_N/C_0 then increases exponentially with N:

$$\langle C_N\rangle/C_0 = (1 + R\varphi)^N\,, \tag{9.39}$$

at a rate per round

$$\frac{1}{N}\ln\frac{\langle C_N\rangle}{C_0} = \ln(1 + R\varphi)\,. \tag{9.40}$$

Since this rate is positive and increases monotonically with φ, it is maximized by choosing $\varphi = 1$. (When $R < 0$, the rate is negative and is minimized by $\varphi = 0$: do not bet on a losing game!) But this maximum is pathological, because it implies betting one's entire capital at every round. Only a single sequence of outcomes is successful: winning all N rounds in a row so that the capital grows dramatically to $C_N/C_0 = 2^N$, but with the tiny probability $((1 + R)/2)^N$. Every other sequence loses, sooner or later, with $C_N = 0$; so the mean is given by (9.40). Thus, the mean is not a useful measure of the capital distribution from which to select φ in any sensible way.

More promising is to examine (following Maslov et al. (1998), and also discussed by Ethier (2004)) the *median* of the distribution, labeled C_N^* and defined by

$$\left\langle \Theta\left(C_N - C_N^*\right)\right\rangle = 1\,/\,2. \tag{9.41}$$

To determine C_N^*, differentiate (9.41) with respect to φ, so that

$$\frac{\partial C_N^*}{\partial \varphi} = N\, C_N^* \frac{\partial}{\partial \varphi} \langle \ln\left(1+\varphi\Omega\right)\rangle\,; \tag{9.42}$$

reintegrating gives

$$\frac{1}{N} \ln \frac{C_N^*}{C_0} = \langle \ln\left(1+\varphi\Omega\right)\rangle\,. \tag{9.43}$$

Thus, the median capital also varies exponentially with N; but now the rate per round, the right-hand side of (9.43), has a vanishing first derivative at an intermediate value of φ. Since its second derivative there is negative definite, the value corresponds to a maximum. For $R^2 \ll 1$ the maximum is found at $\varphi \ll 1$ and so, by expanding (9.43) to second order in φ and R, is given by $\varphi_{\text{opt}} \approx R/\sigma^2 > 0$. The median capital, evaluated at that maximum, grows as $(C_N^*/C_0)_{\text{opt}} \approx \exp(N\, R^2/2\sigma^2)$.

The approach of Kelly (and of most of the subsequent literature) to selecting φ is instead described as "maximizing the expectation of the rate of capital growth." Although that terminology is loose mathematically – growth cannot be ascribed to a stochastic variable, only to some measure over its distribution – the usual derivation leads to the same results as here. Our criterion, though, of maximizing the (exponential) growth rate of the median of the distribution, seems more precise and better motivated.

Note, though, that the growth rate of the median turns negative (i.e., the median shrinks) for fractions larger than about $2R$, despite the mean being positive; the rate heads toward $-\infty$ as the fraction tends toward one. The larger bets increase the risk of losing streaks eating into Player's capital; and as $\varphi \to 1$ Player is ruined very quickly with almost 100% probability. Also notice that the exponential character of the growth is discernible only for $N > \left\langle\!\left\langle \Theta(R)\, R^2/\sigma^2 \right\rangle\!\right\rangle^{-1}$, which might be 10^4 or more; for fewer rounds, the increase still looks linear.

So-called Kelly betting (there is little agreement on terminology in the literature: other labels sometimes used include scaled, proportional, multiplicative, and geometric) seems to have the remarkable advantage that Player is never ruined, in the sense of losing all his money. If he experiences a losing streak, so that his capital shrinks, he is directed merely to reduce his bet sizes accordingly and to continue play. But a bet size that varies continuously toward zero is clearly not allowed in a casino: bets can only be integer multiples of a nonzero table minimum, B_-. Rather, "ruin" should be interpreted as capital so low that the prescribed fractional bet cannot be placed.

Once ruin is reconsidered in this way, analysis of optimal Kelly betting can proceed in analogy with Sect. 9.1. Instead of wins/losses being additive/subtractive to capital, here it's to the *log* of capital. And instead of ruin occurring when $C = 0$, here it's when $C = C_- \equiv B_-/\varphi$. With these analogies, a drifting diffusion equation like (9.34) also applies here, but after modification to the drift coefficients, the replacement of the C_0 variable by $\ln C_0$, and the generalized ruin boundary condition. Thus, the risk satisfies

$$\left[\frac{\partial}{\partial N} - V\frac{\partial}{\partial \ln C_0} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial(\ln C_0)^2}\right] W_N = 0, \tag{9.44}$$

with drift $V \equiv \langle \ln(1 + \varphi\Omega)\rangle$; the same differential equation determines the median capital. The appropriate solutions, at least in the limit of very large N, become

$$W = \exp\left(-2\frac{V}{\sigma^2}\ln\frac{C_0}{C_-}\right), \quad \frac{1}{N}\ln\frac{C_N^*}{C_0} = V(1 - W)\,. \tag{9.45}$$

Again maximizing $\ln\left(C_N^*/C_0\right)$ with respect to φ, but now subject to a fixed risk, leads to the same optimal Kelly fraction as before. The maximal rate, however, is reduced by the survival factor. With V evaluated at the maximum, the risk becomes

$$W_{\text{opt}} = \exp\left[-\frac{R^2}{\sigma^4}\ln\frac{C_0 R}{B_-\sigma^2}\right] \tag{9.46}$$

which, under typical conditions, is extremely small.

Blackjack, however, is a compound game with a varying expected return dependent on the count. In this case, R becomes a random variable, as in Sect. 9.1. A functional maximization of C_N^* with respect to $\varphi(R)$, with $R > R_-$, still gives $\varphi(R)_{\text{opt}} = R/\sigma^2$ as before; the maximum median capital becomes $C_N^* \cong C_0 \exp(N\langle R^2/\sigma^2\rangle/22)$. When R is below a positive threshold the session is terminated in ruin, as discussed above.

Another frequently voiced criticism of Kelly betting is that it leads to wide swings in capital, even though the median is growing at a maximal rate. To reduce the size of the fluctuations about the median, some authors (and a number of practitioners) resort to so-called "fractional Kelly"; here the bet is sized at less than the optimal fraction of the current capital (i.e., "full Kelly"), perhaps half or even a quarter of the optimum.

Just as the median (or equivalently the expected logarithm) of the capital can easily be derived, so the variance of log capital is also readily obtained:

$$\text{var} \equiv \left\langle\left(\ln\frac{C}{C_0}\right)^2\right\rangle - \left\langle\ln\frac{C}{C_0}\right\rangle^2 \cong N\sigma^2\left(\frac{1}{2}\ln\frac{1+\varphi}{1-\varphi}\right)^2 \tag{9.47}$$

which rises monotonically with φ from the origin, initially as φ^2. Thus, a lower bet fraction than the optimum does reduce the variance. At the same time, a measure of return on investment, such as $\langle \ln C/C_0 \rangle / \sqrt{\text{var}}$, increases slowly with decreasing φ; but it does not exhibit a peak, so there is no optimal fraction of Kelly based on such a measure. Kelly bettors are forced to use their own judgment in selecting a reduced Kelly fraction.

On the other hand, criticism of full Kelly based on the size of its capital fluctuations almost always overlooks the concept of ruin in Kelly betting. Should a wide downward swing in capital occur, the player might encounter the ruin threshold and simply start over with a replenished stake, rather than continue to play with a tiny one and even tinier bets. Recovering from a situation of sharply diminished (but not ruined) capital usually takes a dismayingly long playing time, despite the exponentially increasing median.

9.2.2 Multiple Simultaneous Hands

A technique that combats high Kelly variance, while at the same time enhancing yield, involves multiple simultaneous hands. Thus, consider playing H seats simultaneously at the same table, with a capital redistribution technique: initially, Player splits his total capital evenly among the seats, creating subpools, and at each seat bets a uniform fraction φ of the subpool capital. After this and every subsequent round, Player then repools his capital, again divides it evenly (or as evenly as feasible) among the seats, and repeats the betting process. In effect, the scheme uses the winning seats to replenish the losing ones. A scheme of this kind has been proposed for financial investing by Maslov and Zhang (1998).

Let the capital of the hth seat at the end of the nth round be C_n^h, so that the redistributed capital per seat is

$$\overline{C}_n = \frac{1}{H} \sum_{h=1}^{H} C_n^h. \tag{9.48}$$

Then, at that seat, the capital following the next round becomes

$$C_{n+1}^h = \left(1 + \varphi_H \, \Omega_{n+1}^h\right) \overline{C}_n. \tag{9.49}$$

The stochastic process Ω_n^h is uncorrelated from other rounds but, as seen in Sect. 7.1, has a nonvanishing covariance with the other hands on the same round. Thus, after N rounds the redistributed capital per seat has evolved to

$$\overline{C}_N / \overline{C}_0 = \prod_{n=1}^{N} \left(1 + \varphi_H \, \overline{\Omega}_n\right), \tag{9.50}$$

where $\overline{\Omega}_n \equiv H^{-1} \sum_{h=1}^{H} \Omega_n^h$. This is again a multiplicative process, like that for a single seat, and the mean of the redistributed capital is

$$\left\langle \overline{C}_N \right\rangle = \overline{C}_0 \left(1 + R\varphi_H\right)^N . \tag{9.51}$$

On the other hand, the median, $\overline{C}_N^*$, satisfies

$$\frac{1}{N} \ln \frac{\overline{C}_N^*}{\overline{C}_0} = \frac{1}{N} \sum_{n=1}^{N} \left\langle \ln \left(1 + \varphi_H \overline{\Omega}_n\right) \right\rangle. \tag{9.52}$$

Anticipating that the median rate's maximum occurs for $\varphi \ll 1$, $R \ll 1$, expand in this limit:

$$\begin{aligned} \frac{1}{N} \ln \frac{\overline{C}_N^*}{\overline{C}_0} &= \frac{1}{N} \sum_{n=1}^{N} \left(\varphi_H \left\langle \overline{\Omega}_n \right\rangle - \frac{1}{2} \varphi_H^2 \left\langle \overline{\Omega}_n^2 \right\rangle + \cdots \right) \\ &= \varphi_H R - \frac{1}{2} \varphi_H^2 \left[\sigma^2 + (H-1)\,\Gamma\right] / H + \cdots \end{aligned} \tag{9.53}$$

Thus, the maximum rate now occurs for the proportionality fraction $\varphi_H = H\,R / \left(\sigma^2 + (H-1)\,\Gamma\right) \approx H\,R / \left(1.26 + .47\,(H-1)\right)$, where the effect of the higher order terms in (9.53) can be shown to be negligible. At this fraction,

$$\frac{1}{N} \ln \frac{\overline{C}_N^*}{\overline{C}_0} = \frac{H\,R^2 / 2}{\sigma^2 + (H-1)\,\Gamma}. \tag{9.54}$$

Remarkably, both the maximal rate of growth of the median capital and the betting fraction that achieves it have increased as a result of playing H hands simultaneously and redistributing capital among them, as compared with Kelly betting with the total capital at a single seat. As seen in (9.53), the single-hand variance is reduced by a factor of H, which in turn boosts the growth rate, although the effect is partially offset by the covariance. Thus, in the context of Kelly betting, playing multiple hands and systematically redistributing capital improve performance vs. that of just a single hand with the same total capital.

The log capital variance in this scheme, for small but otherwise arbitrary values of φ_H, is var $= N\left[\sigma^2 + (H-1)\,\Gamma\right] \varphi_H^2 / H$; it too would increase, proportionally, if the optimal bet fraction of the previous paragraph were employed. But since typical numerical values are $\Gamma \approx 0.47 < \sigma^2 \approx 1.26$, the coefficient of φ_H^2 in the variance expression decreases with increasing H. As a result, if Player maintains the same bet fraction as if playing only a single hand, namely $\varphi_1 = R / \sigma^2$, rather than the larger value from the optimality criterion, then playing several hands would both reduce the variance as well as raise the median's rate of increase! In this technique, the median capital and the variance become

$$\ln\left(\frac{\overline{C}_N^*}{C_0}\right) = \frac{N\,R^2}{2\sigma^2}\left[1 + \frac{H-1}{H}\left(1 - \frac{\Gamma}{\sigma^2}\right)\right],$$

$$\text{var} = \frac{NR^2}{\sigma^2}\left[1 - \frac{H-1}{H}(1 - \frac{\Gamma}{\sigma^2})\right].$$

Even with only two seats, the median rate is raised by more than 30% while the variance is reduced by the same percentage; with still more simultaneous hands the factors can exceed 50%.

9.3 Back-Counting and Table-Hopping

Section 4.3 has described the playing technique of table-hopping and its variant, back-counting. These maneuvers offer a significant improvement in yield, quantified in Table 4.5. Here the analysis underlying the table (following Werthamer (2006) and (2008)) is described for entry, exit, and departure, singly and in combination.

9.3.1 Entry

Begin with certain assumptions about Player and the table he is at:

1. The game is played with more than one deck, $D > 1$, so that the expected return R_0 for the first round after a shuffle is negative and has variance σ^2.
2. The cards are reshuffled after a shoe penetration F.
3. Player is tracking the dealt cards using one of the usual balanced counting methods, such that the true count immediately after a shuffle is $\gamma = 0$ and the expected return becomes positive, $R > 0$, for true counts greater than a cross-over, $\gamma_0 > 0$.
4. Entry is allowed between any successive rounds, not just at shuffles.

The back-counter enters the game only when the true count first reaches the threshold $\gamma_E \geq \gamma_0$. Thus, we need the probability of this occurring, conditional on the true count being zero immediately following a shuffle. But this probability is mathematically congruent to the probability of ruin, i.e., of the player's capital C first reaching zero from its initial value of C_0. Equation 9.6 has shown that the probability of ruin first occurring at round N, with a bet B per round, is

$$L_N = (C_0/NB)\, p_N(0), \tag{9.55}$$

where (9.18)

$$p_N(C) \equiv \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left[-\frac{(C - C_0 - NBR)^2}{2NB^2\sigma^2}\right] \tag{9.56}$$

is the distribution of capital after N rounds, in the absence of ruin. Furthermore, L_N can be shown to satisfy the drifting diffusion equation (like (9.34))

$$\left(\frac{\partial}{\partial N} - BR\frac{\partial}{\partial C_0} - \frac{1}{2}B^2\sigma^2\frac{\partial^2}{\partial C_0^2}\right) L_N = 0. \tag{9.57}$$

In parallel, the distribution of true counts was shown ((8.24) and (8.7), with $\Lambda = 1$) to be

$$\rho(\gamma, \tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{\gamma^2}{2\tau}\right), \tag{9.58}$$

contingent on the initial condition of zero true count at zero penetration. As a modification from the notation of Chap. 8, the parameter $\tau \equiv 52f/D(1-f) = (52\Delta)^2$ is used; the extra factor of 52 appears because the true count is defined as running count *per deck* undealt. We call τ "time" because the distribution can be verified to satisfy the diffusion equation

$$\left(\frac{\partial}{\partial\tau} - \frac{1}{2}\frac{\partial^2}{\partial\gamma^2}\right) \rho(\gamma, \tau) = 0, \tag{9.59}$$

where τ plays the mathematical role that time does in the physical diffusion process. (The fact that τ is a nonlinear function of depth, which *does* increase proportionally to real time, reflects the character of the underlying random card-dealing process: the true count, which is always zero just after a shuffle and then random walks, must return to zero if/when all cards in the shoe are dealt. Such a constrained process is sometimes called a "Brownian bridge.")

Thus, there is a close analogy in their evolution between the distributions of capital and true count (compare (9.57) and (9.59)), which gives the probability of the true count first reaching γ_E at "time" τ_E as the analog of (9.55),

$$\rho_1(\gamma_E, \tau_E) = (\gamma_E/\tau_E)\ \rho(\gamma_E, \tau_E). \tag{9.60}$$

Furthermore, the distribution $\rho_E(\gamma, \tau)$ of true count γ at any "time" τ prior to entry must still satisfy (9.59) but now with the boundary condition $\rho_E(\gamma_E, \tau) = 0$. This boundary condition is the analog of that for "survival," from Sect. 9.1; it is satisfied by including a "reflection" about γ_E:

$$\rho_E(\gamma, \tau) = (\rho(\gamma, \tau) - \rho(2\gamma_E - \gamma, \tau))\, \Theta(\gamma_E - \gamma), \tag{9.61}$$

where Θ is again the unit step function. Note that any solution to the diffusion equation (and many other differential equations) is unique if it also satisfies the initial/boundary conditions.

This pair of expressions is further justified by showing that they satisfy conservation of probability. Thus, the total probability of *having* entered at any τ_E prior to τ is, from (9.60),

$$\tilde{P}_E(\tau) = \int_0^\tau d\tau_E\, \rho_1(\gamma_E, \tau_E) = 2\,\mathrm{Erf}\left(\frac{\gamma_E}{\sqrt{\tau}}\right), \tag{9.62}$$

whereas the total probability of *not* having entered prior to τ is, from (9.61),

$$P_E(\tau) = \int_{-\infty}^{\infty} d\gamma\, \rho_E(\gamma, \tau) = 1 - 2\,\mathrm{Erf}\left(\frac{\gamma_E}{\sqrt{\tau}}\right). \tag{9.63}$$

These indeed sum to unity, as required by conservation of probability.

In addition, the two expressions are also linked by the iterative "chain rule,"

$$\frac{1}{\tau_E} \int_0^{\tau_E} d\tau \int_{-\infty}^{\infty} d\gamma\, \rho_1\left(\gamma_E - \gamma, \tau_E - \tau\right)\, \rho_E\left(\gamma, \tau\right) = \rho_1\left(\gamma_E, \tau_E\right); \tag{9.64}$$

the integration steps needed for its proof also serve as a prototype for similar convolutions in succeeding sections, so are described in Appendix 2.

Next, given that the true count γ_E is first reached at τ_E, the subsequent evolution of the true count distribution is again a solution of (9.59) with those initial conditions, specifically $\rho\left(\gamma - \gamma_E, \tau - \tau_E\right)$, a generalization of (9.58). Then the distribution of true counts experienced by a back-counter who bets only *after* the true count has first reached γ_E is the convolution with the probability (9.60),

$$\int_0^{\tau} d\tau_E\, \rho\left(\gamma - \gamma_E, \tau - \tau_E\right)\, \rho_1\left(\gamma_E, \tau_E\right) = \rho\left(\Gamma_E, \tau\right), \tag{9.65}$$

$$\Gamma_E \equiv |\gamma - \gamma_E| + \gamma_E\,;$$

performing the integration uses the same methods as in Appendix 2. The back-counter, once he is entered, then bets an amount $B(\gamma)$ based on the true count, and plays until the next reshuffle at penetration F. With the true count distribution of (9.66), his yield is

$$Y_E = \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\ R(\gamma)\ P\left(\gamma\right), \quad P(\gamma) \equiv \frac{1}{F} \int_0^{F} df\ \rho\left(\Gamma_E, \tau\right). \tag{9.66}$$

The entry threshold γ_E is chosen so as to maximize the yield. The distribution $P(\gamma)$ is graphed in Fig. 9.8.

9.3.2 Entry and Exit

Thus far we have assumed that the back-counter, once he has entered the game, remains in until the shoe is reshuffled. But some authorities suggest (e.g., Vancura and Fuchs 1998, p. 132) that he leave the table – we will call it "exit" – when the true count drops below the roughly $+1$ value at which the expected return crosses from positive to negative. We will analyze the more general situation of exit at a true count γ_X, and choose it jointly with γ_E to maximize the yield.

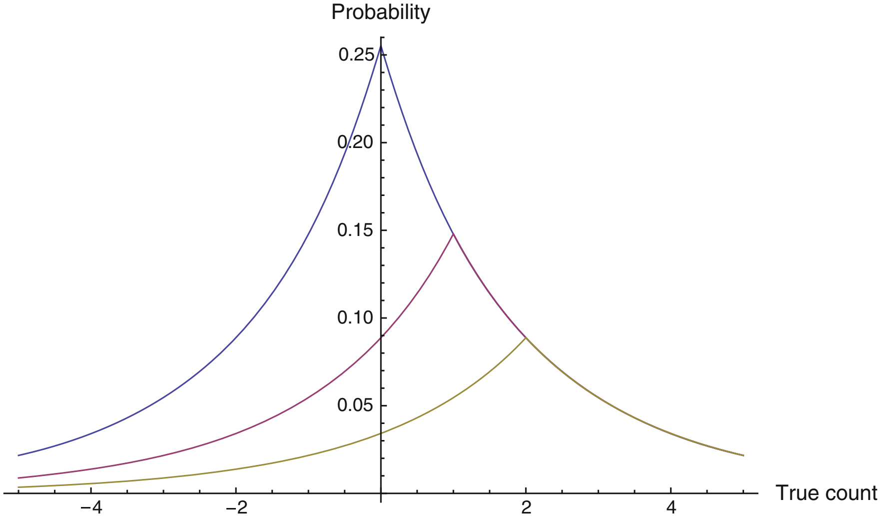

Fig. 9.8 Distributions of true counts, subsequent to entry at true counts 0, 1, 2

Without exit, the distribution of true counts following entry is given by (9.69), and satisfies the diffusion equation, (9.59). With exit, the modified distribution $\rho_{EX}(\gamma, \tau)$ for $\gamma \geq \gamma_X$ at time τ must still satisfy (9.59) but now with the boundary condition $\rho_{EX}(\gamma_X, \tau) = 0$. The solution, analogous to (9.61), is accomplished with a reflection about γ_X:

$$\rho_{EX}(\gamma, \tau) = [\rho(\Gamma_E, \tau) - \rho(|2\gamma_X - \gamma - \gamma_E| + \gamma_E, \tau)]\, \Theta(\gamma - \gamma_X)\,. \qquad (9.67)$$

Equations (9.67) and (9.66) are compared schematically in Fig. 9.9, for the same game parameters as in Fig. 9.8.

Furthermore, the probability of exiting at "time" τ_X, conditional on entry at τ_E, is the convolution of their respective probabilities,

$$\begin{aligned} P_X(\tau_X) &= \int_0^{\tau_X} d\tau_E\; \rho_1(\gamma_X - \gamma_E, \tau_X - \tau_E)\; \rho_1(\gamma_E, \tau_E) \\ &= \rho_1(2\gamma_E - \gamma_X, \tau_X)\,. \end{aligned} \qquad (9.68)$$

These two expressions, like (9.60) and (9.61) above, similarly satisfy conservation of probability but now conditional on entry:

$$\int_{\tau_E}^{\tau} d\tau_X\; P_X(\tau_X) + \int_{-\infty}^{\infty} d\gamma\; \rho_{EX}(\gamma, \tau) = 2\mathrm{Erf}\left(\frac{\gamma_E}{\sqrt{\tau}}\right), \qquad (9.69)$$

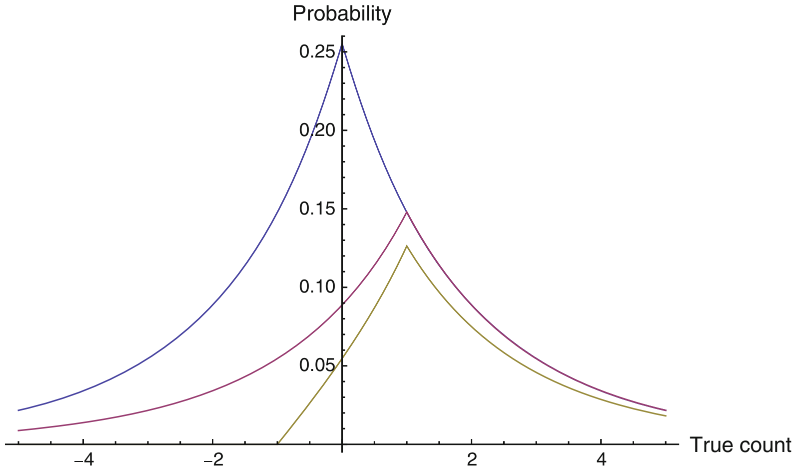

Fig. 9.9 Distributions of true counts: with neither entry nor exit, with entry at $+1$ but without exit, and with entry at $+1$ and exit at -1

where the right-hand side, by (9.62), is the probability of entry prior to τ.

The yield from this table is the analog of (9.66):

$$Y_{EX}^{(1)} = \frac{1}{F}\int_0^F df \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\; R(\gamma)\; \rho_{EX}\,(\gamma,\tau). \tag{9.70}$$

But upon exit, a back-counter can "table-hop" to a second table with a freshly shuffled shoe and repeat the entry process. Hence, the most appropriate assessment of his yield is to include the cash flow from the second table during those rounds of the first that follow the exit and precede its reshuffle. Thus, the yield from the second table alone, weighting with the probability of exit from the first, is

$$Y_{EX}^{(2)} = \int_0^{\tau_F} d\tau_X\, P_X\,(\tau_X)\,\frac{1}{F}\int_0^{F-f_X} df \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\; R(\gamma)\; \rho_{EX}\,(\gamma,\tau). \tag{9.71}$$

By combining the yields from the two tables, and carrying out the intermediate τ_X integration, the total yield becomes

$$\begin{aligned} Y_{EX} &= Y_{EX}^{(1)} + Y_{EX}^{(2)} \\ &= \frac{1}{F}\int_0^F df\left[1 + 2\,\mathrm{Erf}\left(\frac{2\gamma_E - \gamma_X}{\sqrt{\bar{\tau}}}\right)\right]\int_{-\infty}^{\infty} d\gamma\, B(\gamma)R(\gamma)\rho_{EX}(\gamma,\tau). \end{aligned} \tag{9.72}$$

where $\bar{\tau} \equiv 52\,(F - f)/D\,(1 - F + f)$. Although contributions from additional tables beyond the second should in principle be included, in practice these are negligible: the probability of exit from the second table is quite small and the number of rounds played at the third is typically too few to generate much additional value. The optimal entry and exit thresholds are determined, as in the entry-only case, by jointly maximizing the total yield.

9.3.3 Entry and Departure

Separately from a possible exit *following* entry, the table-hopper may independently choose to "depart" the table *prior* to entry. This might occur, for example, if much of the shoe has been dealt without reaching the entry threshold; or if the true count becomes decidedly negative and the probability of it swinging sufficiently positive to trigger entry is correspondingly low. Ideally, the departure decision should be based on a combination of these two circumstances. But Player decision-making predicated on both true count and depth parameters together seems difficult; rather, we consider departure based only on true count.

Exit, occurring after entry by definition, imposes the single boundary constraint $\gamma \ge \gamma_X$, satisfied as per (9.67) by a reflection about γ_X. Departure, occurring alternatively to entry, instead imposes the two simultaneous boundary constraints $\gamma_D \le \gamma \le \gamma_E$, which requires the more complex structure of an infinite array of reflections. A real-world example is that if one looks in a mirror one sees a single reflection; but if one instead looks in a mirror with a second one parallel to and facing it, one sees infinitely many, receding reflections.

Thus, the solution of the diffusion equation subject to the two boundary constraints is the generalization of (9.61),

$$\begin{aligned} \rho_{ED}(\gamma,\tau) &= \sum_{m=-\infty}^{\infty} [\rho\,(\gamma - a_m) - \rho\,(\gamma - 2\gamma_E - a_m)], \\ a_m &\equiv 2m\,(\gamma_E - \gamma_D)\,. \end{aligned} \tag{9.73}$$

Then the probability of entry at τ_E is given by the generalization of the chain rule, (9.64),

$$\begin{aligned} \rho_{1D}\,(\gamma_E,\tau_E) &= \frac{1}{\tau_E}\int_0^{\tau_E} d\tau \int_{\gamma_D}^{\gamma_E} d\gamma\, \rho_1\,(\gamma_E - \gamma, \tau_E - \tau)\,\rho_{ED}\,(\gamma,\tau) \\ &= \frac{\gamma_E}{\tau_E}\rho\,(\gamma_E,\tau_E) - \frac{\gamma_E - \gamma_D}{\tau_E}\sum_{m=1}^{\infty}[\rho\,(a_m - \gamma_E,\tau_E) \\ &\quad -\rho\,(a_m + \gamma_E,\tau_E)]; \end{aligned} \tag{9.74}$$

the integrations use the same methods as for the chain rule, although the algebra is lengthier. Since the terms in the sum decrease exponentially with increasing m, a reasonable approximation is to keep only the larger of the two $m = 1$ terms; the result is

$$\rho_{1D}\left(\gamma_E, \tau_E\right) \cong \frac{\gamma_E}{\tau_E}\rho\left(\gamma_E, \tau_E\right) - \frac{\gamma_E - \gamma_D}{\tau_E}\rho\left(2\gamma_D - \gamma_E, \tau_E\right). \tag{9.75}$$

The corresponding result for the distribution of γ at τ subsequent to entry, generalizing (9.66), is

$$\begin{aligned}
\rho_{ED}\left(\gamma, \tau\right) &= \int_0^\tau d\tau_E\, \rho\left(\gamma - \gamma_E, \tau - \tau_E\right) \rho_{1D}\left(\gamma_E, \tau_E\right) \\
&= \rho\left(\Gamma_E, \tau\right) - \sum_{m=1}^{\infty} \frac{\gamma_E - \gamma_D}{a_m - \gamma_E}\left[\rho\left(|\gamma - \gamma_E| + a_m - \gamma_E, \tau\right)\right. \\
&\quad \left. -\rho\left(|\gamma + \gamma_E| + a_m + \gamma_E, \tau\right)\right] \\
&\cong \rho\left(\Gamma_E, \tau\right) - \frac{\gamma_E - \gamma_D}{\gamma_E - 2\gamma_D}\rho\left(\Gamma_E - 2\gamma_D, \tau\right).
\end{aligned} \tag{9.76}$$

Then the yield from the first table is

$$Y_{ED}^{(1)} = \frac{1}{F}\int_0^F df \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\, R(\gamma)\, \rho_{ED}\left(\gamma, \tau\right). \tag{9.77}$$

Additionally, the probability of departure is the reverse of (9.74) and (9.75),

$$\begin{aligned}
P_D\left(\tau_D\right) &= \frac{-\gamma_D}{\tau_D}\rho\left(\gamma_D, \tau_D\right) - \frac{\gamma_E - \gamma_D}{\tau_D}\sum_{m=1}^{\infty}\left[\rho\left(a_m + \gamma_D, \tau_D\right)\right. \\
&\quad \left. -\rho\left(a_m - \gamma_D, \tau_D\right)\right] P_D\left(\tau_D\right) \\
&\cong \frac{-\gamma_D}{\tau_D}\rho\left(\gamma_D, \tau_D\right) - \frac{\gamma_E - \gamma_D}{\tau_D}\rho\left(2\gamma_E - \gamma_D, \tau_D\right).
\end{aligned} \tag{9.78}$$

Then the yield from a second table following departure becomes, like (9.71)

$$Y_{ED}^{(2)} = \int_0^{\tau_F} d\tau_D\, P_D\left(\tau_D\right) \frac{1}{F}\int_0^{F - f_D} df \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\, R(\gamma)\, \rho_{ED}\left(\gamma, \tau\right) \tag{9.79}$$

and like (9.72) the total yield from both tables is

$$\begin{aligned}Y_{ED} &= Y_{ED}^{(1)} + Y_{ED}^{(2)} \\ &\cong \frac{1}{F}\int_0^F df \left[1 + 2\mathrm{Erf}\left(\frac{-\gamma_D}{\sqrt{\bar{\tau}}}\right) - 2\frac{\gamma_E - \gamma_D}{2\gamma_E - \gamma_D}\mathrm{Erf}\left(\frac{2\gamma_E - \gamma_D}{\sqrt{\bar{\tau}}}\right)\right] \\ &\quad \times \int_{-\infty}^{\infty} d\gamma\, B(\gamma) R(\gamma)\ \rho_{ED}\left(\gamma, \tau\right).\end{aligned} \tag{9.80}$$

As with exit, the optimal entry and departure thresholds are determined by joint maximization of the yield.

9.3.4 Entry, Exit, and Departure

When the table-hopper employs both departure (prior to entry) and exit (subsequent to entry), the probability distribution of true counts at τ combines (9.67) and (9.76),

$$\begin{aligned}\rho_{EDX}\left(\gamma, \tau\right) &= \int_0^\tau d\tau_E [\rho\left(\gamma - \gamma_E, \tau - \tau_E\right) \\ &\quad -\rho\left(2\gamma_X - \gamma - \gamma_E, \tau - \tau_E\right)]\rho_{1D}\left(\gamma_E, \tau_E\right) \\ &= \left[\rho_{ED}\left(\gamma, \tau\right) - \rho_{ED}\left(2\gamma_X - \gamma, \tau\right)\right]\Theta\left(\gamma - \gamma_X\right),\end{aligned} \tag{9.81}$$

and the yield from the first table is

$$Y_{EDX}^{(1)} = \frac{1}{F}\int_0^F df \int_{-\infty}^{\infty} d\gamma\, B(\gamma)\, R(\gamma)\ \rho_{EDX}\left(\gamma, \tau\right). \tag{9.82}$$

The probability of departure remains independent of any possibility of later exit, so the departure contribution to the second table's yield is still given by (9.79); but the probability of exit is influenced by the possibility of previous departure, so that (9.69) generalizes to

$$\begin{aligned}P_{XD}\left(\tau_X\right) &= \int_0^{\tau_X} d\tau_E\, \rho_1\left(\gamma_X - \gamma_E, \tau_X - \tau_E\right)\rho_{1D}\left(\gamma_E, \tau_E\right) \\ &\cong \rho_1\left(2\gamma_E - \gamma_X, \tau_X\right) \\ &\quad -\frac{\gamma_E - \gamma_D}{2\gamma_E - \gamma_D}\rho_1\left(2\gamma_E - \gamma_X - 2\gamma_D, \tau_X\right).\end{aligned} \tag{9.83}$$

Thus, the total yield from the first two tables combined, generalizing (9.72) and (9.80), is

$$
\begin{aligned}
Y_{EDX} = \frac{1}{F} \int_0^F df \Bigg\{ 1 + 2 \left[\operatorname{Erf}\left(\frac{-\gamma_D}{\sqrt{\tau}} \right) + \operatorname{Erf}\left(\frac{2\gamma_E - \gamma_X}{\sqrt{\tau}} \right) \right] \\
- 2 \frac{\gamma_E - \gamma_D}{2\gamma_E - \gamma_D} \left[\operatorname{Erf}\left(\frac{2\gamma_E - \gamma_D}{\sqrt{\tau}} \right) + \operatorname{Erf}\left(\frac{2\gamma_E - \gamma_X - 2\gamma_D}{\sqrt{\tau}} \right) \right] \Bigg\} \\
\times \int_{-\infty}^{\infty} d\gamma \, B(\gamma)\, R(\gamma)\, \rho_{EDX}(\gamma, \tau)\,.
\end{aligned}
\tag{9.84}
$$

As before, the three threshold true counts are determined jointly by maximizing the yield. Computations with (9.84), despite its apparent complexity, are feasible and examples of the resulting thresholds are listed in Table 4.5.

Appendix 1 The Risk of Ruin Formula

Concentrate here on the expression for risk of ruin, (9.12), which is slightly simpler than that for mean capital, (9.9), although similar methodologies apply to both. Begin by combining (9.12) with its sequential predecessors (9.6), (9.4), and (9.1). Then use the two Kronecker deltas to eliminate all but one of the three n_ω summations; the most convenient variable to remain is $\lambda \equiv n_+ + n_-$. Next, proceed to the asymptotic limit of large N as in Appendix 1 of Chap. 8: replace all factorials by their asymptotic limit (the Stirling approximation, $n! \approx \sqrt{2\pi n}(n/e)^n$); change the remaining two sums over discrete variables into integrations over continuous ones; and convert the integration over μ (seen in (9.12)) by the Method of Stationary Phase, with limits extended to $\pm\infty$, into an easily performed Gaussian. This leaves only the last integration, in the form

$$
\begin{aligned}
W_N &\approx \int_{c_0}^{N} d\lambda \frac{c_0 \exp E(\lambda)}{\sqrt{2\pi\lambda\left(\lambda^2 - c_0^2\right)}}, \\
E(\lambda) &\equiv \sum_{\pm} \frac{\lambda \mp c_0}{2} \ln \frac{2\lambda\xi(\pm)}{\sigma^2 (\lambda \mp c_0)}.
\end{aligned}
\tag{9.85}
$$

Next, apply the Stationary Phase method again: the maximum of $E(\lambda)$ with respect to λ is at $\lambda_0 \equiv c_0/|\hat{R}|$, where $\hat{R} \equiv (\xi(+) - \xi(-))/\sigma^2$; so that in the asymptotic limit it becomes $E(\lambda) \approx E_0 - \hat{R}^2(\lambda - \lambda_0)^2/2\lambda$, with $E_0 \equiv -c_0\Theta(\hat{R}) \ln \frac{1+\hat{R}}{1-\hat{R}}$. Thus, W_N asymptotically becomes

$$
W_N \approx \int_0^{\hat{N}} d\lambda \frac{c_0}{\sqrt{2\pi\lambda^3}} \exp\left(E_0 - \frac{\hat{R}^2}{2\lambda}\left(\lambda - \frac{c_0}{|\hat{R}|}\right)^2 \right). \tag{9.86}
$$

Although this integral is not itself a Gaussian, it nevertheless can be evaluated as a closed form: it can be looked up in tables or in the *Mathematica* software package, or converted into a sum of Gaussian integrals by changing the integration variable to $y = \left(c_0 - |\hat{R}|\lambda\right) \Big/ \sqrt{\lambda}$. The result is (9.12) in the text, QED.

Appendix 2 Chain Rule Convolution

Begin with the expression for the ratio,

$$J \equiv \frac{1}{\tau_E} \int_0^{\tau_E} d\tau \int_{-\infty}^{\infty} d\gamma\, \rho_1 \left(\gamma_E - \gamma, \tau_E - \tau\right) \rho_E(\gamma, \tau) / \rho_1 \left(\gamma_E, \tau_E\right). \tag{9.87}$$

After substitution from (9.60) and (9.58), the integrand can be rearranged into the form

$$\begin{aligned} J = \frac{1}{\gamma_E} \int_{-\infty}^{\gamma_E} d\gamma \int_0^{\tau_E} d\tau \sqrt{\frac{(\gamma_E - \gamma)^2 \tau_E}{2\pi(\tau_E - \tau)^3 \tau}} &\left\{ \exp\left[-\frac{\left(\frac{\gamma}{\tau} - \frac{\gamma_E}{\tau_E}\right)^2}{2\left(\frac{1}{\tau} - \frac{1}{\tau_E}\right)} \right] \right. \\ &\left. - \exp\left[-\frac{\left(\frac{2\gamma_E - \gamma}{\tau} - \frac{\gamma_E}{\tau_E}\right)^2}{2\left(\frac{1}{\tau} - \frac{1}{\tau_E}\right)} \right] \right\}. \end{aligned} \tag{9.88}$$

Next change integration variable: for the first term in the braces substitute $u = \gamma^2 \left(\tau^{-1} - \tau_E^{-1}\right)$, while for the second term substitute $u' = (2\gamma_E - \gamma)^2 \left(\tau^{-1} - \tau_E^{-1}\right)$. Then

$$\begin{aligned} J = \frac{1}{\gamma_E} \int_{-\infty}^{\gamma_E} d\gamma &\left\{ \int_0^{\infty} du \frac{|\beta|}{\sqrt{2\pi u^3}} \exp\left(-\frac{(u + \beta)^2}{2u} \right) \right. \\ &\left. - \int_0^{\infty} du' \frac{|\beta'|}{\sqrt{2\pi u'^3}} \exp\left(-\frac{(u' + \beta')^2}{2u'} \right) \right\}, \end{aligned} \tag{9.89}$$

where $\beta \equiv (\gamma - \gamma_E)\,\gamma / \tau_E$, $\beta' \equiv (\gamma_E - \gamma)\,(2\gamma_E - \gamma) / \tau_E$. But

$$\int_0^{\infty} du \frac{|\beta|}{\sqrt{2\pi u^3}} \exp\left(-\frac{(u + \beta)^2}{2u} \right) = \exp\left(-\beta - |\beta|\right) \tag{9.90}$$

as listed, e.g., in *Mathematica* or derivable via the further change of variable $v = (u + \beta) \big/ \sqrt{u}$. Finally, shift the γ integration to $\tilde{\gamma} = -\gamma$ for the first term in the braces and to $\tilde{\gamma}' = \gamma_E - \gamma$ for the second, revealing a major cancellation between the two terms while the uncancelled part of the $\tilde{\gamma}$ integral becomes trivial. The result is just $J = 1$, QED.